Shadows of Black Holes:
A Computational Approach

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Abstract

One of the recent breakthroughs in astronomy was the first observation of a black hole shadow, due to gravitational light deflection in the vicinity of a black hole photon sphere. This thesis sets out to visualize gravitational light deflection by a Schwarzschild black hole using Python simulations. Starting from basics of differential geometry, the concepts and definitions of metric, connection, Riemann curvature tensor and the geodesic equation are introduced. Then moving to Einstein's general theory of relativity, the field equations are stated and the Schwarzschild metric is found as the unique static spherically symmetric solution. In order to derive the geodesics of Schwarzschild, which are crucial in providing the theoretical basis for this computational project, the Euler-Lagrange equations are applied. Thus, the ordinary differential equations (ODE) describing the trajectories of light rays near the black hole are obtained. Next, the initial value problem of the given ODE can be solved, yielding the trajectories of light rays approaching the black hole for given initial conditions. Python simulation algorithms are presented in pseudo-code form, which include the algorithms computing the trajectories, and the visualization of light deflection. Finally, the visualization algorithm reveals how rectangular coordinates are transformed to show how equirectangular images as input are optically distorted by the Schwarzschild black hole.

Contents

Chapter 1

Introduction

One of the most recent astronomical achievements was the discovery of a black hole shadow created by gravitational light deflection in the vicinity of a black hole photon sphere [2–4].

A ring-like form with a dark core area [4], the black hole's shadow, was discovered using a variety of calibration and imaging techniques, and it stayed consistent over numerous EHT observations. As visible in Figure 1.1, Messier 87 [2], a massive galaxy in the nearby Virgo galaxy cluster, features a black hole at its center.

Figure 1.1 First Shadow [3]

The Event Horizon Telescope (EHT) is a planet-scale array of eight ground-based radio telescopes that was created in conjunction with scientists from all around the world to capture images of a black hole [2]. Which is the basis for a technique known as very-long-baseline interferometry (VLBI) [2]. VLBI synchronizes telescopes all around the world and uses the rotation of our planet to form a gigantic, Earth-size observatory [2].

On April 10, 2019, EHT researchers revealed that they have succeeded in presenting the first direct visual evidence of a supermassive black hole and its shadow [2–4]. As predicted by Einstein's general relativity, the EHT is the result of years of international work and gives scientists with a new instrument for exploring the Universe's most extreme objects.

Chapter 2

Spacetime

2.1 Differential Geometry

In this section, the basic differential geometry concepts and definitions involved in this signature work, including tensor, metric, connection, Riemann metric, and geodesics, are given with the help of *Advanced General Relativity* written by John Stewart [1].

2.1.1 Tensor

Definition 1. *A* $(1,2)$ *tensor* S *on* $T_p(M)$ *is a map* [1]

$$S : T_p(M) \times T_p(M) \times T_p^*(M) \to R$$

which is linear in each argument.

2.1.2 Metric

Definition 2. *A metric tensor* g *at a point* p *in* M *is a symmetric* $(0,2)$ *tensor. It assigns a magnitude* $\sqrt{|g(X,X)|}$ *to each vector* X *in* $T_p(M)$*, denoted by* $d(X)$*, and defines the angle between any two vectors* X*,* Y *of* *non-zero magnitude in* $T_p(M)$ *via* [1]

$$a(X,Y) = \arccos\left[\frac{g(X,Y)}{d(X)\,d(Y)}\right].$$

2.1.3 Connection

Definition 3 (Linear Connection [1])**.** *A linear connection* ∇ *on* M *is a map sending every pair of smooth vector fields* (X,Y) *to a vector field* $\nabla_X Y$ *such that*

$$\nabla_X (aY + Z) = a\nabla_X Y + \nabla_X Z$$

for any constant scalar a*, but*

$$\nabla_X (fY) = f\nabla_X Y + (Xf)\,Y$$

when f *is a function, and it is linear in* X

$$\nabla_{X+fY} Z = (\nabla_X Z + f\nabla_Y Z).$$

Further, acting on functions f, ∇X is defined by

$$\nabla_X f = Xf.$$

Definition 4 (Covariant Derivative)**.** $\nabla_X Y$ *is not linear in* Y*,* ∇ *is not a tensor. While,* $\nabla_X Y$ *is linear in* X*, thus defining a* $(1,1)$ *tensor* ∇Y *mapping* X *to* $\nabla_X Y$*, called the covariant derivative of* Y*, where* $\nabla_X Y$ *is called the covariant derivative of* Y *with respect to* X*.*

Definition 5 (Components of Connection [1])**.** *Let* (e_a) *be a basis for vector fields and write* ∇_{e_a} *as* ∇_a*. Since* $\nabla_a e_b$ *is a vector there exist scalars* ${\Gamma^c}_{ba}$ *such that*

$$\nabla_a e_b = {\Gamma^c}_{ba} e_c.$$

The $\Gamma^c{}_{ba}$ *are called the components of the connection.*

Theorem 1. *If a manifold possesses a metric* g *then there is a unique symmetric connection, the Levi-Civita connection or metric connection* ∇ *such that* [1]

$$\nabla g = 0.$$

Moreover, the Levi-Civita connection coefficient in the three-dimensional Euclidean space shares the same definition with Christoffel symbols in Riemannian geometry, which is

Definition 6. *Christoffel symbols is defined by formula* [1]

$$\Gamma^i{}_k m = \frac{1}{2} g^{in} \left(g_{mn,k} + g_{kn,m} - g_{km,n} \right).$$

2.1.4 Riemann

The metric tensor g, describes what the Levi-Civita connection, which in return completely describes Riemann curvature tensor. The Riemann curvature tensor in which the gravitational field is actually manifests. That is, the metric tensor g is said to describe the gravitational field.

Definition 7. *Riemann metric tensor is* [1]

$$R\left(X,Y\right)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{|X,Y|} Z,$$

where ∇_X*,* ∇_Y *and* $\nabla_{|X,Y|}$ *are Levi-Civita connection.*

Definition 8. *Riemann curvature tensor is* [1]

$$R\left(u,v\right)w = \nabla_u \nabla_v w - \nabla_v \nabla_u w - \nabla_{[u,v]} w.$$

Moreover, the Ricci curvature tensor is a contraction of the 1st and 3rd index of Riemann curvature tensor, which is defined as

Definition 9. *The Ricci scalar is* [1]

$$R = g^{ab} R_{ab}.$$

Definition 10. *The Ricci curvature tensor is* [1]

$$R_{ab} = R^{c}_{acb}.$$

2.1.5 Geodesic Equation

Generally, starting from the metric tensor which generalizes the property of Euclidean space, a manifold is called Riemannian manifold if it is equipped with positive definite metric tensor. On a Riemannian manifold, the curve connecting two points, that has the smallest length is called geodesic.

Definition 11 (Geodesics [1])**.** *Let X be a vector field such that $\nabla_X X = 0$. Then the integral curves of X are called geodesics.*

Theorem 2 ([1])**.** *There is precisely one geodesic through a given point $p \in M$ in a given direction X_p.*

2.2 Einstein's Field Equations

The basic form of Einstein's field equations is given as [1]

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

where $R_{\mu\nu}$ represents Ricci curvature tensor, R represents Ricci curvature scalar, $g_{\mu\nu}$ represents metric tensor, Λ represents cosmological constant, G represents gravitational constant, c represents the speed of light, $\kappa = 8\pi G/c^4 \approx 2.077 \times 10^{-43} N^{-1}$ represents the Einstein gravitational constant, and $T_{\mu\nu}$ represents the stress energy-momentum tensor. The left hand side of the equation, $\left(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu}\right)$, tells the geometry of spacetime, by showing the curvature of spacetime as determined by the metric g. While the right hand side of the equation, $\left(\frac{8\pi G}{c^4}T_{\mu\nu}\right)$, displays matter energy content of spacetime. Or frankly speaking, the right hand side describes the matter movement. Hence, easily speaking, the the reason why Einstein's field equations are important is it connects geometry of spacetime with matter movement.

2.2.1 Components of Einstein's Field Equations

The spacetime described by Einstein's field equation is measured in four dimensions which are 0 time, 1 x-axis, 2 y-axis, and 3 z-axis, denoted by the Greek letters $\mu\nu$. Where 1, 2, and 3, i.e., x, y, and z-axis together describe the space while 0 describes the time. In this case, though the Einstein's field equation seems like to be a single equation, there are actually 16 non-linear partial differential equations expended by the single equation. The reason is that because the spacetime described by the equation has four dimensions, which results in $R_{\mu\nu}$, $g_{\mu\nu}$, $g_{\mu\nu}$ and $T_{\mu\nu}$, these four tensors, all being in 4 dimensions. Therefore, there are $4 \times 4 = 16$ equations in total. However, due to the symmetry property of tensors, 6 equations out of 16 are duplicate, which makes them total of 10 non-linear partial differential equations. The expended metric tensor from g_{00} to g_{33} is shown as

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{01} & g_{02} & g_{03} \\ g_{10} & g_{11} & g_{12} & g_{13} \\ g_{20} & g_{21} & g_{22} & g_{23} \\ g_{30} & g_{31} & g_{32} & g_{33} \end{pmatrix}.$$

Since the metric tensor is a symmetric tensor, that is to say, $g_{\mu\nu} = g_{\nu\mu}$, the ten components of the metric tensor without the duplicate ones are therefore

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{01} & g_{02} & g_{03} \\ & g_{11} & g_{12} & g_{13} \\ & & g_{22} & g_{23} \\ & & & g_{33} \end{pmatrix},$$

which are in total 10 components. Similarly, the stress energy-momentum tensor, $T_{\mu\nu}$, is also a symmetric tensor. That is to say, $T_{\mu\nu} = T_{\nu\mu}$ holds as well. Therefore, as the metric tensor, the components of the stress energy-momentum tensor are in total 10 components, without the duplicate ones.

2.2.2 Ricci Curvature Tensor

As introduced above, the Ricci curvature tensor is a contraction of the 1st and 3rd index of Riemann curvature tensor. Ricci curvature tensor actually tracks volume change along geodesics, which means how volume grow and shrink in geodesics.

Depending on the curvature, which is actually the manifold that are dealing with, Ricci curvature R shows the change in volume. And it happens either in a static way, a growing or decreasing way. Since the volume change is not so relevant to this signature work, it will not be further discussed.

2.2.3 Ricci Scalar

As shown above, Ricci curvature tensor is actually a measurement how an object shrinks or grows in size, or remains static based on the sign of curvature of spacetime ($= 0$ static, > 0 converging, < 0 diverging).

While, Ricci scalar actually shows how an object deviates from standard Euclidean space. And in this case, the sign of Ricci scalar really matters.

The definition of Ricci scalar is $R = g^{ab}R_{ab}$ [1].

2.2.4 Einstein Tensor

Since Einstein's gravity is with the curvature of spacetime and Riemannian geometry which dealing with curvature, based on Marcel Grossman and Michele Besso's basic thoughts of tensor, Einstein curvature tensor is given as a combination of Ricci curvature tensor and metric tensor, defined as

Definition 12. *The Einstein curvature tensor is* [1]

$$G_{ab} = R_{ab} - \frac{1}{2}Rg_{ab},$$

which can also be derived directly from Einstein's field equations.

2.2.5 Stress Energy-Momentum Tensor

The stress energy-momentum tensor is actually the relativistic extension of classical stress tensor. It describes energy and momentum flux throughout spacetime and gives rise to the gravitational field in general relativity.

As shown in Einstein's field equations, the stress energy-momentum tensor can be expressed as

$$T_{\mu\nu} = \begin{pmatrix} T_{00} & T_{01} & T_{02} & T_{03} \\ T_{10} & T_{11} & T_{12} & T_{13} \\ T_{20} & T_{21} & T_{22} & T_{23} \\ T_{30} & T_{31} & T_{32} & T_{33} \end{pmatrix},$$

where T_{00} describes energy density, $T_{01}, T_{02}, T_{03}, T_{10}, T_{20}, T_{30}$ describe momentum density, $T_{12}, T_{21}, T_{13}, T_{31}, T_{23}, T_{32}$ describe shear stress, and T_{11}, T_{22}, T_{33} describe pressure.

Chapter 3

Schwarzschild

Now we have the Einstein's field equations and we are able to see how they predict the existence of black holes, gravitational waves and the expansion of the universe by effecting on cosmology.

When it comes to black hole solutions, we can categorize them for electrically charged and uncharged black holes, and rotating and non-rotating black holes. We are going to look at the uncharged non-rotating case, called the Schwarzschild solution, which is governed by the Schwarzschild metric. The Schwarzschild metric is the spacetime metric for a spherically symmetric non-rotating mass that has no electric charge.

The Schwarzschild metric predicts gravitational time dilation, the gravitational Doppler effect, the bending of light due to gravity, shifting in perihelion of orbits, and the existence of black holes with event horizon with a radius of R_s, which is also called the Schwarzschild radius [5].

The metric was first discovered by Karl Schwarzschild, who published the paper "On the Gravitational Field of a Mass Point according to Einstein's Theory" in Janurary, 1916, only few months after Einstein's general relativity paper in 1915.

3.1 Schwarzschild Metric

In this section, we will show basic derivation of Schwarzschild metric [5].

3.1.1 Coordinates

Since we are dealing with a spherically symmetric mass, it is better to deal with spacetime in spherical coordinates. Therefore, we are going to use the spherical coordinates (ct, r, θ, ϕ) instead of the Cartesian coordinates (ct, x, y, z) as defined

$$
\begin{aligned}
r &= \sqrt{x^2 + y^2 + z^2} \\
\theta &= \arccos\left(\frac{z}{r}\right) = \arccos\left(\frac{z}{\sqrt{x^2 + y^2 + z^2}}\right) \\
\phi &= \arctan\left(\frac{y}{x}\right),
\end{aligned}
$$

where r is the radius, θ is the angle from the north pole or called co-latitude, and ϕ is the angle of rotation around the vertical axis or called longitude, as shown in Figure 3.1.

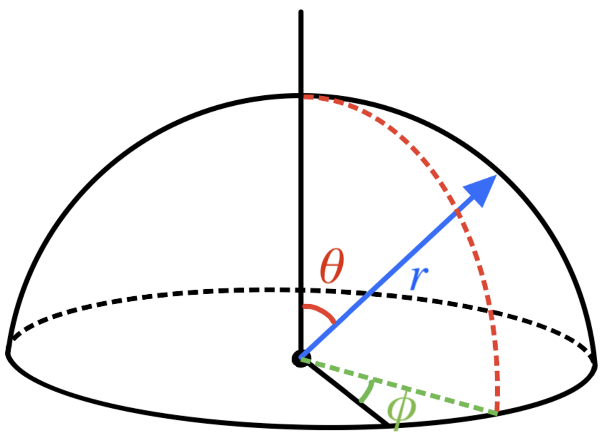


Figure 3.1 Spherical Coordinates

3.1.2 Summations

Greek letters are used as summation index, which refers to all four spacetime indices

$$g_{\mu\nu},\ \mu\nu : 0 \to ct, 1 \to r, 2 \to \theta, 3 \to \phi.$$

While, Latin or English letters are used for the summation index which only refers to the spatial indices without time, as

$$g_{ij},\ ij : 1 \to r, 2 \to \theta, 3 \to \phi.$$

3.1.3 Schwarzschild Metric Derivation

As introduced above, given that the basic form of Einstein's field equations is given as [1]

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}.$$

To solve for the Schwarzschild solution from the Einstein's field equations, basically, we put in an energy-momentum tensor $T_{\mu\nu}$ to the right hand side of the Einstein's field equations and solve for the metric tensor $g_{\mu\nu}$, where the Ricci curvature tensor $R_{\mu\nu}$ is made of the second derivative of $g_{\mu\nu}$. That is to say, we put in a description of the mass energy and momentum in spacetime, and we get out a complete description of the geometry of spacetime.

The energy-momentum tensor of the interior of a body is non-zero. For example, inside our earth, there are mass and pressure such that the energy-momentum tensor inside the earth is non-zero. However, the exterior of a body, lets say, outside the earth, we can assume that the spacetime is approximately a vacuum so that the energy-momentum tensor $T_{\mu\nu}$ is zero.

Thus, for a spherically symmetric mass, we set the energy-momentum tensor $T_{\mu\nu}$ in the Einstein's field equations to be zero.

Moreover, we are also going to set the cosmological constant to zero since it is basically negligible unless we are working at cosmological scales, which gives $\Lambda = 0$.

Therefore, the components $\frac{8\pi G}{c^4}T_{\mu\nu}$ and $\Lambda g_{\mu\nu}$ are all zero and what is left is actually the Einstein tenor

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 0.$$

If we take the trace of what is left using the inverse metric tensor, $g^{\mu\nu}$, as

$$G_{\mu\nu}g^{\mu\nu} = R_{\mu\nu}g^{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}g^{\mu\nu} = 0,$$

where the trace of the Ricci tensor is the Ricci scalar, i.e., $R_{\mu\nu}R^{\mu\nu} = R^{\mu}_{\mu}$, and the trace of $g_{\mu\nu}g^{\mu\nu}$ is the four by four identity matrix δ^{μ}_{μ} whose value is 4.
Thus we have

$$R^{\mu}_{\mu} - \frac{1}{2}R\delta^{\mu}_{\mu} = R - \frac{1}{2}R \cdot 4 = 0$$

which ends up with

$$R = 0,$$

i.e., the Ricci scalar is zero.

Hence, for a vacuum region, the Einstein's field equations is simplified to the Ricci tensor being zero, as $R^{\mu}_{\mu} = 0$, which is called the Ricci flat spacetime [5]. Given the spacetime is Ricci flat, there are no immediate changes in the volume of a group of test particles outside the massive particle.

While, the vacuum outside the mass still involves curved spacetime because the Riemann curvature tensor here is non-zero, as $R^{\rho}_{\sigma\mu\nu} \neq 0$.

Therefore, the Schwarzschild metric derivation takes $R^{\mu}_{\mu} = 0$ and solves for the components of the 4×4 spacetime metric, that describes the curved spacetime near a massive particle, M.

Moreover, the assumption that the effects of gravity become negligible as moving far away from the mass M is proposed. That is to say, the spacetime becomes basically flat described by the Minkowski metric in Cartesian coordinates,

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

As for the Minkowski metric in spherical coordinates which represents the flat spacetime, given the Cartesian coordinates basis as

$$\begin{aligned} x &= r\left(\sin\theta\right)\left(\cos\phi\right) \\ y &= r\left(\sin\theta\right)\left(\sin\phi\right) \\ z &= r\left(\cos\theta\right), \end{aligned}$$

we need to change from Cartesian coordinates basis to the spherical coordinates basis using multi-variable chain rule,

$$\begin{aligned} \frac{\partial}{\partial r} &= \frac{\partial x}{\partial r}\frac{\partial}{\partial x} + \frac{\partial y}{\partial r}\frac{\partial}{\partial y} + \frac{\partial z}{\partial r}\frac{\partial}{\partial z} \\ \frac{\partial}{\partial \theta} &= \frac{\partial x}{\partial \theta}\frac{\partial}{\partial x} + \frac{\partial y}{\partial \theta}\frac{\partial}{\partial y} + \frac{\partial z}{\partial \theta}\frac{\partial}{\partial z} \\ \frac{\partial}{\partial \phi} &= \frac{\partial x}{\partial \phi}\frac{\partial}{\partial x} + \frac{\partial y}{\partial \phi}\frac{\partial}{\partial y} + \frac{\partial z}{\partial \phi}\frac{\partial}{\partial z}, \end{aligned}$$

which gives

$$\begin{aligned} \frac{\partial}{\partial r} &= \sin\theta\cos\phi\frac{\partial}{\partial x} + \sin\theta\sin\phi\frac{\partial}{\partial y} + \cos\theta\frac{\partial}{\partial z} \\ \frac{\partial}{\partial \theta} &= r\cos\theta\cos\phi\frac{\partial}{\partial x} + r\cos\theta\sin\phi\frac{\partial}{\partial y} - r\sin\theta\frac{\partial}{\partial z} \\ \frac{\partial}{\partial \phi} &= -r\sin\theta\sin\phi\frac{\partial}{\partial x} + r\sin\theta\cos\phi\frac{\partial}{\partial y} + 0, \end{aligned}$$

then calculate the dot products of the basis vectors,

$$\frac{\partial^2}{\partial r^2} = \frac{\partial}{\partial r} \cdot \frac{\partial}{\partial r} = -\sin^2\theta\cos^2\phi - \sin^2\theta\sin^2\phi - \cos^2\theta$$
$$\frac{\partial^2}{\partial \theta^2} = \frac{\partial}{\partial \theta} \cdot \frac{\partial}{\partial \theta} = -r^2\cos^2\theta\cos^2\phi - r^2\cos^2\theta\sin^2\phi - r^2\sin^2\theta$$
$$\frac{\partial^2}{\partial \phi^2} = \frac{\partial}{\partial \phi} \cdot \frac{\partial}{\partial \phi} = -r^2\sin^2\theta\sin^2\phi - r^2\sin^2\theta\cos^2\phi.$$

We are able to get these components for the metric tensor in spherical coordinates,

$$\frac{\partial^2}{\partial r^2} = -1$$
$$\frac{\partial^2}{\partial \theta^2} = -r^2$$
$$\frac{\partial^2}{\partial \phi^2} = -r^2\sin^2\theta,$$

while

$$\frac{\partial^2}{\partial ct^2} = +1$$

which is unchanged.

Hence, the Minkowski metric in spherical coordinates is given as

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2\sin^2\theta \end{bmatrix}.$$

The Minkowski metric should be the metric far away from the mass as r approaches infinity and spacetime becomes flat. While close to the mass, the metric components of the curved spacetime are unknown. However, some assumptions can be used to narrow down the exact form that the metric components should take.

The first assumption is we assume that the spacetime is static, which tells two things [5],

1. the metric does not depend on time, that is to say

$$\frac{\partial}{\partial t} g_{\mu\nu} = 0;$$

2. the spacetime is symmetric when reversing the time coordinate, which is equivalent to say that $g_{\mu\nu}$ will not change as $t \to -t$. This also implies that the black hole is non-rotating.

Moreover, since basis vectors are just partial derivatives with respect to a coordinate variable (here is r, θ, ϕ), reversing the direction of the time coordinate also reverses the direction of the time basis vector, as

$$e_t = \frac{\partial}{\partial ct} \to \frac{\partial}{\partial c\,(-t)} = -\frac{\partial}{\partial ct} = -e_t$$

as $t \to -t$.
However, for different components of the metric, the sign will vary, as

$$g_{tt} = g\,(e_t, e_t) = g\,(-e_t, -e_t) = g_{tt}$$

while

$$g_{ti} = (e_t, e_i) = g\,(-e_t, e_i) = -g_{ti},$$

where $i = r,\ \theta,\ \phi$.
Since $g_{ti} = -g_{ti}$, for $i = r,\ \theta,\ \phi$ we have

$$g_{ti} = 0,$$

that is to say, the spacetime metric $g_{\mu\nu}$'s components g_{tr}, $g_{t\theta}$, $g_{t\phi}$, g_{rt}, $g_{\theta t}$,

$g_{\phi t}$ are all 0, which gives to be

$$\begin{bmatrix} g_{tt} & 0 & 0 & 0 \\ 0 & g_{rr} & g_{r\theta} & g_{r\phi} \\ 0 & g_{\theta r} & g_{\theta\theta} & g_{\theta\phi} \\ 0 & g_{\phi r} & g_{\phi\theta} & g_{\phi\phi} \end{bmatrix}.$$

The second assumption made is that the spacetime is spherical symmetry, which is equivalent to say that the θ and ϕ components should resemble the metric for a sphere of radius r, that is,

$$\begin{bmatrix} g_{tt} & 0 & 0 & 0 \\ 0 & g_{rr} & g_{r\theta} & g_{r\phi} \\ 0 & g_{\theta r} & g_{\theta\theta} & g_{\theta\phi} \\ 0 & g_{\phi r} & g_{\phi\theta} & g_{\phi\phi} \end{bmatrix} \Rightarrow \begin{bmatrix} g_{tt} & 0 & 0 & 0 \\ 0 & g_{rr} & g_{r\theta} & g_{r\phi} \\ 0 & g_{\theta r} & -C(r)\, r^2 & 0 \\ 0 & g_{\phi r} & 0 & -C(r)\, r^2 \sin^2\theta \end{bmatrix}$$

where $C(r)$ is radial scaling function.

Moreover, if we want the radial basis vector e_r to stick out normal to the sphere in the radial direction, it must be perpendicular to both e_θ and e_ϕ, which gives

$$e_\theta \cdot e_r = 0 = g_{\theta r} = g_{r\theta}$$

$$e_\phi \cdot e_r = 0 = g_{\phi r} = g_{r\phi}.$$

And hence, under our assumption so far, the spacetime metric is diagonal and is given as

$$\begin{bmatrix} g_{tt} & 0 & 0 & 0 \\ 0 & g_{rr} & 0 & 0 \\ 0 & 0 & -C(r)\, r^2 & 0 \\ 0 & 0 & 0 & -C(r)\, r^2 \sin^2\theta \end{bmatrix},$$

where the remaining g_{tt} and g_{rr} components should only depend on the radial coordinate r, if we want to maintain spherical symmetry as stated early above.

Let g_{tt} to be $A(r)$ and g_{rr} to be $B(r)$. Since the metric component $B(r)$ corresponds to a space-like direction (follow the $(+---)$ equation), using a negative sign for it to get

$$\begin{bmatrix} A(r) & 0 & 0 & 0 \\ 0 & -B(r) & 0 & 0 \\ 0 & 0 & -C(r)\,r^2 & 0 \\ 0 & 0 & 0 & -C(r)\,r^2\sin^2\theta \end{bmatrix}.$$

For further simplification, we redefine the radial coordinate r to be $\widetilde{r} = \sqrt{C(r)}r$ which will eliminate the function $C(r)$ on the last two metric components and gives the metric as

$$\begin{bmatrix} A(\widetilde{r}) & 0 & 0 & 0 \\ 0 & -B(\widetilde{r}) & 0 & 0 \\ 0 & 0 & -\widetilde{r}^2 & 0 \\ 0 & 0 & 0 & -\widetilde{r}^2\sin^2\theta \end{bmatrix}$$

and just for simplicity, just write $\widetilde{r}$ as r in the following context, thus, the Schwarzschild metric $g_{\mu\nu}$ is given in the form of

$$\begin{bmatrix} A(r) & 0 & 0 & 0 \\ 0 & -B(r) & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2\sin^2\theta \end{bmatrix}.$$

After simplifying the form of the metric as much as possible, in order to solve for $A(r)$ and $B(r)$, we will going to calculate the Levi-Civite connection coefficients $\Gamma^{\sigma}_{\mu\nu}$, then calculate the Ricci tensor $R_{\mu\nu}$ and then force the metric to give us the results of Newtonian gravity in the limit of low velocity and weak gravity, which will give us the Schwarzschild metric.

There are 13 non-zero Levi-Civite connection coefficients in the Schwarzschild solution, and only 9 of them are independent.

Let us start by calculating the Levi-Civite connection coefficients.

Since the spacetime metric $g_{\mu\nu}$ is diagonal, we can easily get the inverse metric just by taking the reciprocal of all of the diagonal elements as

$$g^{\mu\nu} = \begin{bmatrix} \frac{1}{A(r)} & 0 & 0 & 0 \\ 0 & \frac{1}{-B(r)} & 0 & 0 \\ 0 & 0 & \frac{1}{-r^2} & 0 \\ 0 & 0 & 0 & \frac{1}{-r^2\sin^2\theta} \end{bmatrix}$$

whose indices are

$$\begin{aligned} g_{00} &= A(r), & g^{00} &= \frac{1}{A(r)}, \\ g_{11} &= -B(r), & g^{11} & \frac{1}{-B(r)}, \\ g_{22} &= -r^2, & g^{22} & \frac{1}{-r^2}, \\ g_{33} &= -r^2\sin^2\theta, & g^{33} & \frac{1}{-r^2\sin^2\theta}. \end{aligned}$$

Given the standard formula for the Levi-Civite connection coefficients as

$$\Gamma^{\sigma}_{\mu\nu} = \frac{1}{2} g^{\sigma\alpha}\left(\partial_\nu g_{\alpha\mu} + \partial_\mu g_{\alpha\nu} - \partial_\alpha g_{\mu\nu}\right),$$

since the metric is diagonal, the two indices of Levi-Civite connection coefficients always need to match if the components are to be non-zero. So we can replace all the α indices with σ to rewrite the formula for the Levi-Civite connection coefficients as

$$\Gamma^{\sigma}_{\mu\nu} = \frac{1}{2} g^{\sigma\sigma}\left(\partial_\nu g_{\sigma\mu} + \partial_\mu g_{\sigma\nu} - \partial_\sigma g_{\mu\nu}\right).$$

Let us start from the calculation of $\Gamma^0_{\mu\nu}$, and the rest of $\Gamma^1_{\mu\nu}$, $\Gamma^2_{\mu\nu}$ and $\Gamma^3_{\mu\nu}$ are similar.

Substitute σ with 0, we have

$$\Gamma^0_{\mu\nu} = \frac{1}{2} g^{00} \left(\partial_\nu g_{0\mu} + \partial_\mu g_{0\nu} - \partial_0 g_{\mu\nu} \right)$$

The values of μ and ν make differences:

1. When $\mu\nu = 00$,

$$\Gamma^0_{00} = \frac{1}{2} g^{00} \left(\partial_0 g_{00} + \partial_0 g_{00} - \partial_0 g_{00} \right).$$

Since g_{00} does not depend on time, all the time derivative terms, i.e., $\partial_0 g_{00}$, $\partial_0 g_{00}$ and $\partial_0 g_{00}$, go to zero which gives

$$\Gamma^0_{00} = 0.$$

2. When $\mu\nu = ii$ where $i = 1, 2, 3$,

$$\Gamma^0_{ii} = \frac{1}{2} g^{00} \left(\partial_i g_{0i} + \partial_i g_{0i} - \partial_0 g_{ii} \right).$$

Since g_{0i} is not the diagonal element of the metric, it is assigned to 0. Moreover, again, since the metric components does not depend on time, which gives

$$\Gamma^0_{ii} = 0, \text{ where } i = 1, 2, 3.$$

3. When $\mu\nu = ij$ where $i, j = 1, 2, 3$ and $i \neq j$,

$$\Gamma^0_{ij} = \frac{1}{2} g^{00} \left(\partial_j g_{0i} + \partial_i g_{0j} - \partial_0 g_{ij} \right).$$

Similarly, g_{0i}, g_{0j} and g_{ij} are off diagonal which gives

$$\Gamma^0_{ij} = 0, \text{ where } i, j = 1, 2, 3 \text{ and } i \neq j.$$

4. When $\mu\nu = 0i$ where $i = 1, 2, 3$,

$$\Gamma^0_{0i} = \frac{1}{2} g^{00} \left(\partial_i g_{00} + \partial_0 g_{0i} - \partial_0 g_{0i} \right).$$

Similar to above, we have $g_{0i} = 0$ which gives

$$\Gamma^0_{0i} = \frac{1}{2} g^{00} \partial_i g_{00}$$

where $g_{00} = A(r)$ and $g^{00} = \frac{1}{A(r)}$.
Since the function $A(r)$ only depends on r, only the partial r term gives a non-zero result, thus

$$\begin{aligned} \Gamma^0_{01} = \Gamma^0_{10} &= \frac{1}{2} \frac{1}{A} \left(\partial_r A(r) \right), \\ \Gamma^0_{02} = \Gamma^0_{20} &= \frac{1}{2} \frac{1}{A} \left(\partial_\theta A(r) \right) = 0, \\ \Gamma^0_{03} = \Gamma^0_{30} &= \frac{1}{2} \frac{1}{A} \left(\partial_\phi A(r) \right) = 0. \end{aligned}$$

Therefore, the only one connection coefficient left for $\Gamma^0_{\mu\nu}$ is $\Gamma^0_{01} = \Gamma^0_{10} = \frac{1}{2} \frac{1}{A} \left(\partial_r A(r) \right)$.

Just as stated, similar to the calculation of $\Gamma^0_{\mu\nu}$, we can calculate the following non-zero connection coefficients for $\Gamma^1_{\mu\nu}$, $\Gamma^2_{\mu\nu}$ and $\Gamma^3_{\mu\nu}$. And all 9 non-zero Levi-Civite connection coefficients are

$$\Gamma^0_{01} = \Gamma^0_{10} = \frac{1}{2}\frac{1}{A}\left(\partial_r A\left(r\right)\right),$$

$$\Gamma^1_{00} = \frac{1}{2}\frac{1}{B}\left(\partial_r A\left(r\right)\right), \qquad \Gamma^1_{11} = \frac{1}{2}\frac{1}{B}\left(\partial_r B\left(r\right)\right), \qquad \Gamma^1_{22} = -\frac{r}{B\left(r\right)}, \qquad \Gamma^1_{33} = -\frac{r\sin^2\theta}{B\left(r\right)},$$

$$\Gamma^2_{33} = -\sin\theta\cos\theta, \qquad \Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{r},$$

$$\Gamma^3_{13} = \Gamma^3_{31} = \frac{1}{r}, \qquad \Gamma^3_{23} = \Gamma^3_{32} = \cot\theta.$$

Presenting these connection coefficients in arrays makes it more obvious

$$\Gamma^0_{\mu\nu} \to \begin{bmatrix} 0 & \frac{\partial_r A}{\partial 2A} & 0 & 0 \\ \frac{\partial_r A}{\partial 2A} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad \Gamma^1_{\mu\nu} \to \begin{bmatrix} \frac{\partial_r A}{\partial 2B} & 0 & 0 & 0 \\ 0 & \frac{\partial_r B}{\partial 2B} & 0 & 0 \\ 0 & 0 & -\frac{r}{B} & 0 \\ 0 & 0 & 0 & -\frac{r\sin^2\theta}{B} \end{bmatrix},$$

$$\Gamma^2_{\mu\nu} \to \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{r} & 0 \\ 0 & \frac{1}{r} & 0 & 0 \\ 0 & 0 & 0 & -\sin\theta\cos\theta \end{bmatrix}, \quad \Gamma^3_{\mu\nu} \to \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{r} \\ 0 & 0 & 0 & \cot\theta \\ 0 & \frac{1}{r} & \cot\theta & 0 \end{bmatrix}.$$

Given these Levi-Civite connection coefficients, to solve for functions $A\left(r\right)$ and $B\left(r\right)$, we can conduct the calculation of Ricci tensors.
To remember, given the Riemann tensor

$$R^\rho_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\alpha_{\nu\sigma}\Gamma^\rho_{\mu\alpha} - \Gamma^\beta_{\mu\sigma}\Gamma^\rho_{\nu\beta},$$

the Ricci tensor is just the Riemann tensor with its upper and lower middle indices summed together, as

$$R_{\gamma\delta} = R^\mu_{\gamma\mu\delta} = \partial_\mu \Gamma^\mu_{\delta\gamma} - \partial_\delta \Gamma^\mu_{\mu\gamma} + \Gamma^\alpha_{\delta\gamma}\Gamma^\mu_{\mu\alpha} - \Gamma^\beta_{\mu\gamma}\Gamma^\mu_{\delta\beta}.$$

Therefore, the Ricci components are

$$R_{00} = R^{\mu}_{0\mu 0} = \partial_{\mu}\Gamma^{\mu}_{00} - \partial_0\Gamma^{\mu}_{\mu 0} + \Gamma^{\alpha}_{00}\Gamma^{\mu}_{\mu\alpha} - \Gamma^{\beta}_{\mu 0}\Gamma^{\mu}_{0\beta},$$
$$R_{11} = R^{\mu}_{1\mu 1} = \partial_{\mu}\Gamma^{\mu}_{11} - \partial_1\Gamma^{\mu}_{\mu 1} + \Gamma^{\alpha}_{11}\Gamma^{\mu}_{\mu\alpha} - \Gamma^{\beta}_{\mu 1}\Gamma^{\mu}_{1\beta},$$
$$R_{22} = R^{\mu}_{2\mu 2} = \partial_{\mu}\Gamma^{\mu}_{22} - \partial_2\Gamma^{\mu}_{\mu 2} + \Gamma^{\alpha}_{22}\Gamma^{\mu}_{\mu\alpha} - \Gamma^{\beta}_{\mu 2}\Gamma^{\mu}_{2\beta}.$$

Now, take R_{00} as example, for each of the four terms, we need to look at the non-zero connection coefficients and decide what the non-zero coefficients in these summations are, referring to the solved 9 non-zero coefficients.

For the Γ^{μ}_{00} in R_{00}, which has two zero indices on the bottom of the connection coefficient, there is only one single non-zero connection coefficient with two zero indices on the bottom which is $\Gamma^{1}_{00} = \frac{1}{2}\frac{1}{B}\left(\partial_r A\left(r\right)\right)$. Thus, even thought the μ index is technically summed from 0 to 3, only the $\mu = 1$ term stays around.

While, for the second term $\Gamma^{\mu}_{\mu 0}$ of R_{00}, there are actually no connection coefficients with 0 index on the lower right with the other two indices matching. Thus, it goes to zero as $\Gamma^{\mu}_{\mu 0} = 0$.

For the third term $\Gamma^{\alpha}_{00}\Gamma^{\mu}_{\mu\alpha}$, α must be 1 to match the non-zero coefficients.

For the fourth term $\Gamma^{\beta}_{\mu 0}\Gamma^{\mu}_{0\beta}$, β can be 0 or 1.

Hence,

$$\begin{aligned}
R_{00} = R^{\mu}_{0\mu 0} &= \partial_\mu \Gamma^{\mu}_{00} - \partial_0 \Gamma^{\mu}_{\mu 0} + \Gamma^{\alpha}_{00}\Gamma^{\mu}_{\mu\alpha} - \Gamma^{\beta}_{\mu 0}\Gamma^{\mu}_{0\beta} \\
&= \partial_1 \Gamma^{1}_{00} - 0 + \Gamma^{1}_{00}\Gamma^{\mu}_{\mu 1} - \left(\Gamma^{0}_{\mu 0}\Gamma^{\mu}_{00} + \Gamma^{1}_{\mu 0}\Gamma^{\mu}_{01}\right) \\
&= \partial_1 \Gamma^{1}_{00} + \Gamma^{1}_{00}\Gamma^{\mu}_{\mu 1} - \Gamma^{0}_{\mu 0}\Gamma^{\mu}_{00} - \Gamma^{1}_{\mu 0}\Gamma^{\mu}_{01} \\
&\overset{\mu \text{ can be } 0,1,2,3}{=} \partial_1 \Gamma^{1}_{00} + \left(\Gamma^{1}_{00}\Gamma^{0}_{01} + \Gamma^{1}_{00}\Gamma^{1}_{11} + \Gamma^{1}_{00}\Gamma^{2}_{21} + \Gamma^{1}_{00}\Gamma^{3}_{31}\right) \\
&- \left(\Gamma^{0}_{00}\Gamma^{0}_{00} + \Gamma^{0}_{10}\Gamma^{1}_{00} + \Gamma^{0}_{20}\Gamma^{2}_{00} + \Gamma^{0}_{30}\Gamma^{3}_{00}\right) - \left(\Gamma^{1}_{00}\Gamma^{0}_{01} + \Gamma^{1}_{10}\Gamma^{1}_{01} + \Gamma^{1}_{20}\Gamma^{2}_{01} + \Gamma^{1}_{30}\Gamma^{3}_{01}\right) \\
&= \partial_1 \Gamma^{1}_{00} + \Gamma^{1}_{00}\Gamma^{0}_{01} + \Gamma^{1}_{00}\Gamma^{1}_{11} + \Gamma^{1}_{00}\Gamma^{2}_{21} + \Gamma^{1}_{00}\Gamma^{3}_{31} - \Gamma^{1}_{10}\Gamma^{1}_{00} - \Gamma^{1}_{00}\Gamma^{0}_{01} \\
&= \partial_1 \Gamma^{1}_{00} + \Gamma^{1}_{00}\Gamma^{1}_{11} + \Gamma^{1}_{00}\Gamma^{2}_{21} + \Gamma^{1}_{00}\Gamma^{3}_{31} - \Gamma^{1}_{10}\Gamma^{1}_{00} \\
&= \partial_1 \Gamma^{1}_{00} + \Gamma^{1}_{00}\Gamma^{1}_{11} + 2\Gamma^{1}_{00}\Gamma^{2}_{21} - \Gamma^{1}_{10}\Gamma^{1}_{00} \\
&= \partial_r \frac{\partial_r A}{2B} + \frac{\partial_r A}{2B}\frac{\partial_r B}{2B} + 2\frac{\partial_r A}{2B}\frac{1}{r} - \frac{\partial_r A}{2A}\frac{\partial_r A}{2B} \\
&= \partial_r \left(\frac{\partial_r A}{2} B^{-1}\right) + \frac{\partial_r A \partial_r B}{4B^2} + \frac{\partial_r A}{rB} - \frac{(\partial_r A)^2}{4AB} \\
&\overset{\text{Chain rule}}{=} \frac{\partial_r^2 A}{2B} - \frac{\partial_r A \partial_r B}{4B^2} + \frac{\partial_r A}{rB} - \frac{(\partial_r A)^2}{4AB}.
\end{aligned}$$

Remember that the Einstein's field equations suggest that all the components of the Ricci tensors are equal to zero, i.e., $R_{00} = 0$, $R_{11} = 0$ and $R_{22} = 0$.
Thus,

$$R_{00} = \frac{\partial_r^2 A}{2B} - \frac{\partial_r A \partial_r B}{4B^2} + \frac{\partial_r A}{rB} - \frac{(\partial_r A)^2}{4AB} = 0.$$

For further simplification, denote the partial derivative of A and B with respect to r as

$$\partial_r A \to A', \; \partial_r B \to B',$$

and multiply the equation by a common denominator of $4AB^2r$ to get the rid of the denominators, after which we have

$$R_{00} = 2rABA'' - rAA'B' + 4ABA' - rBA'^2 = 0.$$

Similarly, we can get R_{11} and R_{22} as

$$\begin{aligned}
R_{11} = R^{\mu}_{1\mu1} &= \partial_\mu \Gamma^{\mu}_{11} - \partial_1 \Gamma^{\mu}_{\mu1} + \Gamma^{\alpha}_{11}\Gamma^{\mu}_{\mu\alpha} - \Gamma^{\beta}_{\mu1}\Gamma^{\mu}_{1\beta} \\
&= -\partial_1 \Gamma^{0}_{01} - 2\partial_1 \Gamma^{2}_{21} + \Gamma^{1}_{11}\Gamma^{0}_{01} + 2\Gamma^{1}_{11}\Gamma^{2}_{21} - \Gamma^{0}_{01}\Gamma^{0}_{10} - 2\Gamma^{2}_{21}\Gamma^{2}_{12} \\
&= -\frac{\partial_r^2 A}{2A} + \frac{(\partial_r A)^2}{4A^2} + \frac{\partial_r A \partial_r B}{4AB} + \frac{\partial_r B}{rB}
\end{aligned}$$

$$\Rightarrow R_{11} = -2rABA'' + rBA'^2 + rAA'B' + 4A^2B' = 0.$$

And

$$\begin{aligned}
R_{22} = R^{\mu}_{2\mu2} &= \partial_\mu \Gamma^{\mu}_{22} - \partial_2 \Gamma^{\mu}_{\mu2} + \Gamma^{\alpha}_{22}\Gamma^{\mu}_{\mu\alpha} - \Gamma^{\beta}_{\mu2}\Gamma^{\mu}_{2\beta} \\
&= \partial_1 \Gamma^{1}_{22} - \partial_2 \Gamma^{3}_{32} + \Gamma^{1}_{22}\left(\Gamma^{0}_{01} + \Gamma^{1}_{11}\right) - \Gamma^{3}_{32}\Gamma^{3}_{23} \\
&= -\frac{1}{B} + 1 - r\frac{\partial_r A}{2AB} + r\frac{\partial_r B}{2B^2}
\end{aligned}$$

$$\Rightarrow R_{22} = -2ABA + 2AB^2 - rA'B + rAB' = 0.$$

Given all the Ricci components are zero,

$$\begin{aligned}
R_{00} &= 2rABA'' - rAA'B' + 4ABA' - rBA'^2 = 0, \\
R_{11} &= -2rABA'' + rBA'^2 + rAA'B' + 4A^2B' = 0, \\
R_{22} &= -2ABA + 2AB^2 - rA'B + rAB' = 0,
\end{aligned}$$

we are able to solve for functions $A(r)$ and $B(r)$.

Note that

$$R_{00} + R_{11} = 4ABA' + 4A^2B' = 0$$

$$\Rightarrow BA' + AB' = 0$$

which is equivalent to say the partial derivative of AB with respect to r is 0, i.e.,

$$\partial_r (AB) = 0.$$

Which implies that AB is a constant. Denote AB as K.

The value of the constant K is invariable and has nothing to do with

r. Thus, as $r \to y$, i.e., as the Schwarzschild metric approaching to the flat Minkowski metric, the value of K is invariable and equals to the value at the Minkowski metric, i.e., $g_{11}g_{22} = 1 \times 1 = 1$.
Thus, the constant $K = 1$ for all r, implies that

$$B(r) = \frac{1}{A(r)}$$

holds for all r.
Substitute $B = \frac{1}{A} = A^{-1}$ and $B' = \partial_r(A^{-1}) = -\frac{A'}{A^2}$ back to R_{22} formula and we have

$$\begin{aligned}
R_{22} &= -2ABA + 2AB^2 - rA'B + rAB' = 0 \\
\Rightarrow &- 2A\frac{1}{A} + 2A\frac{1}{A^2} - rA'\frac{1}{A} - rA\frac{A'}{A^2} = 0 \\
\Rightarrow &- 2A + 2 - 2rA' = 0 \\
\Rightarrow & rA' = 1 - A \\
\Rightarrow & A(r) = 1 - \frac{k}{r}, \text{ where } k \text{ is a constant} \\
\Rightarrow & B(r) = \frac{1}{A(r)} = \left(1 - \frac{k}{r}\right)^{-1}.
\end{aligned}$$

Currently, we have got the form of the Schwarzschild metric as

$$\begin{bmatrix} 1-\frac{k}{r} & 0 & 0 & 0 \\ 0 & -\left(1-\frac{k}{r}\right)^{-1} & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2\sin^2\theta \end{bmatrix}.$$

We can then solve the constant k by forcing the Schwarzschild metric to reproduce Newtonian gravity in the limit of low velocity and weak gravity, we will skip this part because it is too complicated to discuss here.

Actually, k is usually denoted as R_s, the Schwarzschild radius, or called as the event horizon of the black hole, given as

$$k = \frac{2GM}{c^2},$$

where G is Newton's gravitational constant.

Finally, we have the Schwarzschild metric to be

$$\begin{bmatrix} 1-\frac{R_s}{r} & 0 & 0 & 0 \\ 0 & -\left(1-\frac{R_s}{r}\right)^{-1} & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2\sin^2\theta \end{bmatrix},$$

where $k = \frac{2GM}{c^2}$, G is Newton's gravitational constant, c is the speed of light and M is the mass of the Schwarzschild black hole.

3.2 Geodesics

A massive particle's world line through spacetime can be parameterized by its proper time τ. And the geodesic equation parameterized with proper time parameter τ as [5]

$$\frac{\mathrm{d}^2 x^\sigma}{\mathrm{d}\tau^2} + \Gamma^\sigma_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} = 0.$$

3.2.1 Null Geodesics

Since light beams are massless, they always have a proper time of zero by definition, which means $\tau = 0$ for all light beams by all times. Instead, we parameterize light-like paths by a generic path parameter λ, which gives tangent vectors $\frac{\mathrm{d}}{\mathrm{d}\lambda}$ alone the light world lines.

A light-like geodesic (or called null geodesic) is a geodesic where every tangent vector along the path is light-like, and of course, the rate of change of these tangent vectors is zero. Which gives another geodesic equation that is parameterized with a generic path parameter λ instead of proper time τ, that works for massless particles [5],

$$\frac{\mathrm{d}^2 x^\sigma}{\mathrm{d}\lambda^2} + \Gamma^\sigma_{\mu\nu} \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda} \frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda} = 0.$$

The null geodesic equation is actually 4 separate equations, one for each spacetime coordinate through the σ index, which can refer to any of the four spacetime variables, thus

$$\frac{\mathrm{d}^2 ct}{\mathrm{d}\lambda^2} + \Gamma^t_{\mu\nu} \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda} \frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda} = 0,$$
$$\frac{\mathrm{d}^2 r}{\mathrm{d}\lambda^2} + \Gamma^r_{\mu\nu} \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda} \frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda} = 0,$$
$$\frac{\mathrm{d}^2 \theta}{\mathrm{d}\lambda^2} + \Gamma^\theta_{\mu\nu} \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda} \frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda} = 0,$$
$$\frac{\mathrm{d}^2 \phi}{\mathrm{d}\lambda^2} + \Gamma^\phi_{\mu\nu} \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda} \frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda} = 0.$$

3.2.2 Numerical Solution

Rewrite the Schwarzschild metric as

$$ds^2 = c^2 \left(1 - \frac{R_s}{r}\right) dt^2 - \left(1 - \frac{R_s}{r}\right)^{-1} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2$$

where $R_s = 2GM/c^2$ $(*)$.

Following the Schwarzschild metric above, the following Lagrangian can be considered, noted as L^2, such that

$$L^2 = c^2 \left(1 - \frac{R_s}{r}\right) \dot{t}^2 - \left(1 - \frac{R_s}{r}\right)^{-1} \dot{r}^2 - r^2 \left(\dot{\theta}^2 - \sin^2\theta \dot{\phi}^2\right)$$

where $dt = \dot{t}$ and etc.

For the particle whose mass is zero, such as a photon that are used in the following simulation, we choose an arbitrary affine parameter noted λ

for the universe line of the particle, (while if the particle has mass, then the temporal derivatives can be taken with respect to the proper time parameter τ, the parameter of the universe line of the massive particle).

Given the variational principle as $\delta \int \mathcal{L}^2 d\lambda = 0$, the Euler-Lagrange equations gives

$$\partial_{x^\mu}\mathcal{L}^2 - d_\lambda \partial_{\dot{x}^\mu}\mathcal{L}^2 = 0,$$

where $\dot{x}^\mu = x^\mu/d\lambda$, with $x^\mu \in \{t, r, \theta, \varphi\}$ which can be separated into the equivalent system as following,

$$\begin{aligned}
\left(1-\frac{R_s}{r}\right)\dot{t} &= C_1,\\
\left(1-\frac{R_s}{r}\right)^{-1}\ddot{r}+\frac{R_s c^2}{2r^2}\dot{t}^2-\left(1-\frac{R_s}{r}\right)^{-2}\frac{R_s}{2r^2}\dot{r}^2-r\left(\dot{\theta}^2+\sin^2\theta\dot{\phi}^2\right) &= 0,\\
\ddot{\theta}+\frac{2\dot{r}\dot{\theta}}{r}-\sin\theta\cos\theta\dot{\phi}^2 &= 0,\\
r^2\sin^2\theta\dot{\phi} &= C_2,
\end{aligned}$$

with $C1$ and $C2$ as two constants that remain to be identified.

Since the Schwarzschild metric being, by construction, with spherical symmetry, which shows that the trajectories of the particles are plane or more precisely, are contained in a coordinated hypersurface with equation $\theta = \theta_0$ [5]. Thus, the third equation in the above system admits for particular solution $\theta = \pi/2$. In this case, the above system can be simplified into,

$$\begin{aligned}
\left(1-\frac{R_s}{r}\right)\dot{t} &= C_1,\\
\left(1-\frac{R_s}{r}\right)^{-1}\ddot{r}+\frac{R_s c^2}{2r^2}\dot{t}^2-\left(1-\frac{R_s}{r}\right)^{-2}\frac{R_s}{2r^2}\dot{r}^2-r\dot{\phi}^2 &= 0,\\
r^2\dot{\phi} &= h,
\end{aligned}$$

where $h = r^2\dot{\phi} = C_2$.

Moreover, the second equation can be replaced by $ds^2/d\lambda^2 = 0$, that is,

$$\left(1 - \frac{R_s}{r}\right)\dot{t}^2 - \left(1 - \frac{R_s}{r}\right)^{-1}\dot{r}^2 - r^2\dot{\phi}^2 = 0.$$

Combining the other two equations, we have

$$\dot{r}^2 + \frac{h^2}{r^2}\left(1 - \frac{R_s}{r}\right) = c^2 C_1^2.$$

Then change the variable r by $1/u$, deviate from ϕ which gives

$$\frac{d^2u}{d\phi^2} - \frac{3R_s}{2}u^2 + u = 0,$$

which is the photon trajectory described by an ordinary differential equation, also the starting point of our Schwarzschild black hole simulation.

3.3 Event Horizon

Given that the Schwarzschild radius to be

$$R_s = \frac{2GM}{c^2}.$$

Obviously it is only depends on the mass of the object and some physical constants.

Usually, G is very small while c^2 is quite large such that the Schwarzschild radius is very small for most massive objects. However, when an object is so incredibly massive and dense that its Schwarzschild radius R_s becomes larger than its physical radius r_0, the object is then called Schwarzschild black hole and the Schwarzschild radius is then called the black hole's event horizon.

3.4 Photon Sphere

Given the photon trajectory as

$$\frac{d^2u}{d\phi^2} - \frac{3R_s}{2}u^2 + u = 0.$$

In order to calculate the photon sphere (which is the case that r must be constant), by setting r equals to constant, the derivative of u with respect to ϕ, i.e., $\frac{\mathrm{d}}{\mathrm{d}\phi}u = \frac{\mathrm{d}}{\mathrm{d}\phi}\frac{1}{r}$, therefore equals to zero.

Hence,

$$\frac{\mathrm{d}^2u}{\mathrm{d}\phi^2} = 0.$$

Thus, the radius of photon sphere of the Schwarzschild metric can be given as

$$\frac{3R_s}{2}\frac{1}{r^2} + \frac{1}{r} = 0,$$

$$\Rightarrow r_{sphere} = \frac{3}{2}R_s.$$

Chapter 4

Algorithm

4.1 Trajectory

The trajectory function is aimed to solve the differential equation in spherical coordinate for a static black hole, which allows us to compute the photon trajectory given the distance from the black hole and the initial angular speed.

The core of this function is to resolve the initial value problem for the ODE system,

$$v_0(\phi) = u(\phi)', \quad v_1(\phi) = \frac{3R_s}{2}u(\phi)^2 - u(\phi)$$

with

$$v_0(0) = \frac{1}{D}, \qquad v_1(0) = \frac{1}{D\tan\alpha},$$

by simple iterations.

After detecting that the light ray reaches the set boundary, the iterations will be stopped to determine whether the light ray has been captured by the black hole or not. If it is captured, the corresponding interval of initial angle with given precision will be given as result as well.

The algorithm to achieve the desired goal is shown below,

Input: angle: initial angle, belongs to $[0, 180)$ degree; dist: distance from the observer to the center of black hole;
Output: return r, ϕ, $[\alpha_{\min} - j, \alpha_{\min}]$;
Data: $c = 1$: speed of light in vacuum; $G = 1$: Newton Constant; $M = 1$: Black hole mass; $distance\_max = 1$: a multiple of D to prevent divergence; $dphi = 10 ** (-4)$: ϕ's range $<< 1 * (10^{-}4)$ avoiding differences; $ITERATION = int(3 * \pi/\phi)$: points to be calculated (be aware of some trajectories may exceed a full lap); $Rs = 2 * G * M/c ** 2$: Schwarzschild radius;

```
u = [1/dist] * ITERATION, u1 = 1/(dist* tan(angle))
ITERATION_REEL = 0
for i in range(ITERATION − 1) do
    ITERATION_REEL+ = 1
    u2 = 3/2*Rs*u[i]**2−u[i], u1 = u1+u2*dphi, u[i+1] = u[i]+u1*dphi
    if 1/u[i + 1] <= Rs or 1/u[i + 1] > distance_max * dist then
        Break
    end
end
phi = [phi_initial] * ITERATION_REEL
r = [dist] * ITERATION_REEL
for i in range(ITERATION_REEL − 1) do
    phi[i + 1] = phi[i] + dphi, r[i + 1] = 1/u[i]
end
```

By using the above algorithm, the two cases can be displayed and examined.

First case is by fixing the initial angle and gradually decrease the distance between observer and the black hole and the result is shown by Figure 4.1.

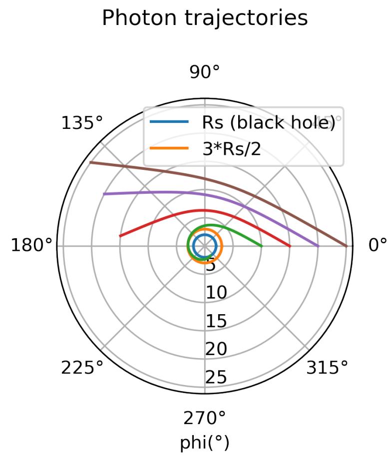


Figure 4.1 Initial angle = 30 degree, initial distance = 10, distance approaching to 0 by 5 each step

While the second case does the opposite, fixing the distance and gradually decrease the initial angle of light rays whose result is shown by Figure 4.2.

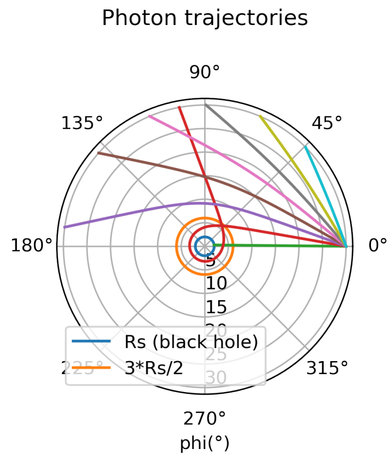


Figure 4.2 Initial angle = 80 degree, initial distance = 30, angle approaching to 0 by 10 degree each step

Moreover, the interval of initial angular speed can be also generated and to examine it, just take the radius of Schwarzschild metric $Rs = 8$, the distance between the observer and black hole $D = 50$ as example, the interval is calculated by the algorithm as $[24.2, 24.3]$ under the precision of 0.1. The algorithm can be set to different precision according to our requirements. As the precision decreases, the interval gets smaller and smaller, as shown in Figure 4.3, allowing us to obtain the interval of initial angle at which light is captured by the black holey, and to use the interval to give the estimated initial angle.

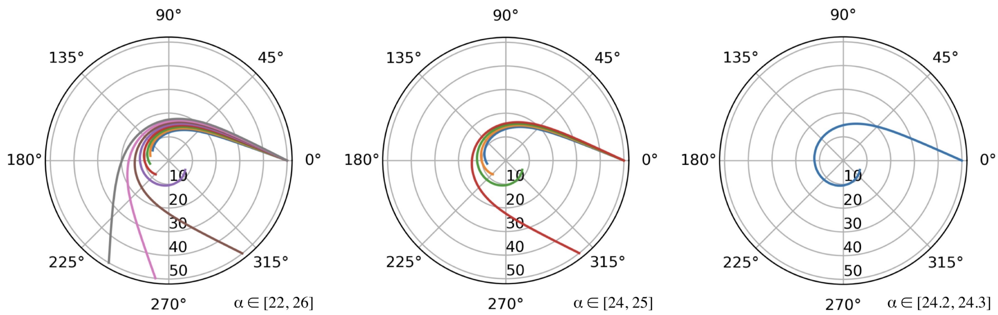


Figure 4.3 Illustrate the $\alpha_{\min}$ search, first precision: $\alpha \in [22, 26]$; second precision: $\alpha \in [24, 25]$, third precision: $\alpha \in [24.2, 24.3]$.

4.2 Visualization

To obtain better visualization of the optical properties of a Schwarzschild black hole, we first need to make coordinate operations, then make interpolation decisions and deal with the pixel colors (R, G, B) information. And a flow chart that displays the whole process has been generated as shown in section 4.2.2.

4.2.1 Coordinate Operations

As shown in the geodesic part, by spherical symmetry, we can restrict to the Schwarzschild equatorial plane without loss of generality which gives,

$$\theta = \frac{\pi}{2},$$
$$\frac{\mathrm{d}\theta}{\mathrm{d}\lambda} = \frac{\mathrm{d}^2\theta}{\mathrm{d}\lambda^2} = 0.$$

That is to say, we can always rotate our coordinate system so that the light trajectories lie in the equatorial plane of $\theta = \pi/2$ as shown in the Figure 4.4.

Given equirectangular images are two-dimensional images, the image distorted by the black hole can be interpreted as the whole equirectangular image rotates inside and outside the image plane with different rotation axes centered on the black hole. Then the light trajectories close to the black hole can then be calculated to complete distorted image.

In this case, the best way to get the resulting vector when rotating the given vector around the axis is by using Euler–Rodrigues formula.

The Euler–Rodrigues formula explains the three-dimensional rotation of a vector. It employs a different parametrization than Rodrigues' rotation formula — the rotation is represented by four Euler parameters due to Leonhard Euler.

Given the three-dimensional vector $\mathbf{v} = [v_x, v_y, v_z]$ that is needed to be rotated around the three-dimensional axis $\mathbf{u} = [u_x, u_y, u_z]$ and the rotation angle θ in radian, the four real numbers a, b, c and d which represents the rotation can be calculated as

$$\begin{aligned} a &= \cos\frac{\theta}{2}, \\ b &= -\frac{\mathbf{u_x}}{\sqrt{\mathbf{u_x}\cdot\mathbf{u_x}}}\sin\frac{\theta}{2}, \\ c &= -\frac{\mathbf{u_y}}{\sqrt{\mathbf{u_y}\cdot\mathbf{u_y}}}\sin\frac{\theta}{2}, \\ d &= -\frac{\mathbf{u_z}}{\sqrt{\mathbf{u_z}\cdot\mathbf{u_z}}}\sin\frac{\theta}{2}. \end{aligned}$$

Then, the rotation matrix α can be calculated as

$$\alpha = \begin{bmatrix} a^2+b^2-c^2-d^2 & 2(bc-ad) & 2(bd+ac) \\ 2(bc+ad) & a^2-b^2+c^2-d^2 & 2(cd-ab) \\ 2(bd-ac) & 2(cd+ab) & a^2-b^2-c^2+d^2 \end{bmatrix}.$$

Hence, the vector $\mathbf{v}'$ which is rotated by the rotation of vector $\mathbf{v}$ around axis $\mathbf{u}$ in θ angle can then be represented as

$$\mathbf{v}' = \alpha\mathbf{v}.$$

Once the rotation matrix α are generated and finished the whole rotation on equirectangular image, it is pretty easy to conduct the following coordinate operations. We then need to calculate the final position of light rays according to the previous trajectory algorithm, and then transform the light rays' final position back to the equirectangular image plane using the inverse rotation matrix $-\alpha$.

Finally, loops go through each pixel to assign the (x_2, y_2) pixel color (R, G, B) information to the (x, y) pixel.

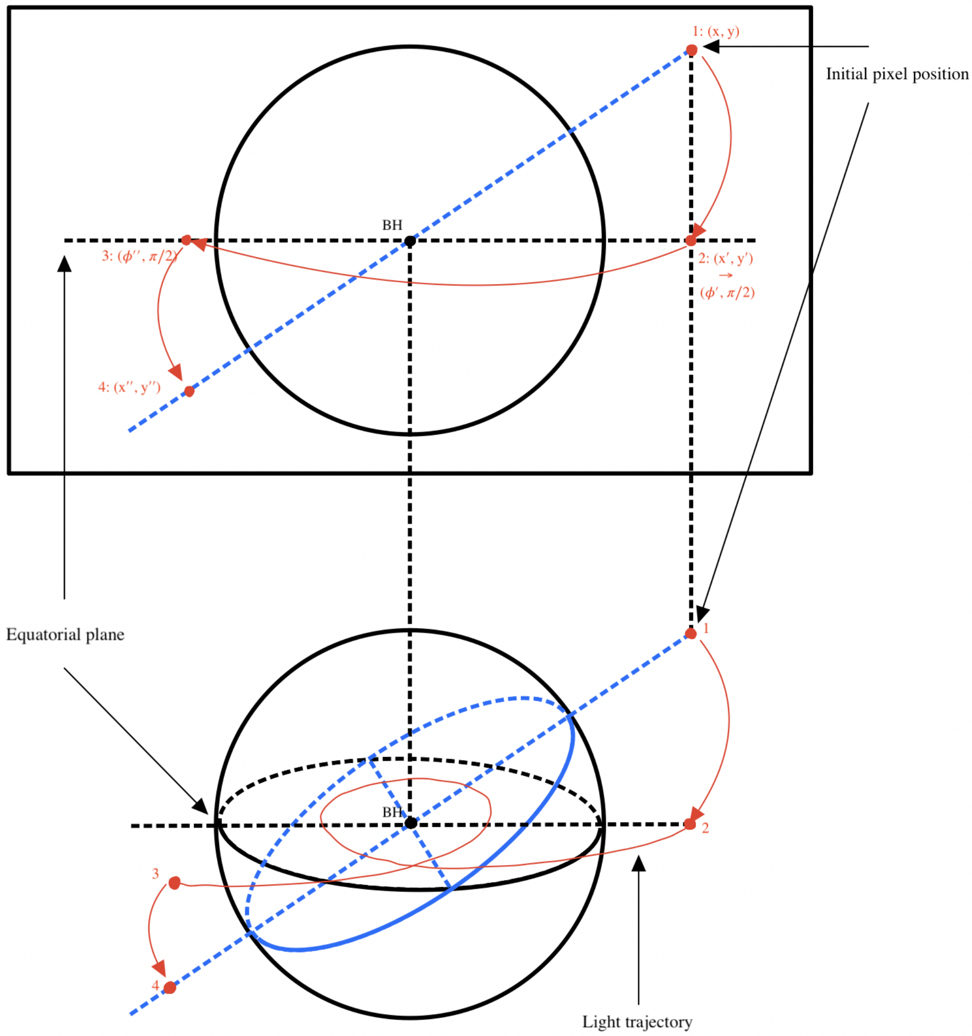
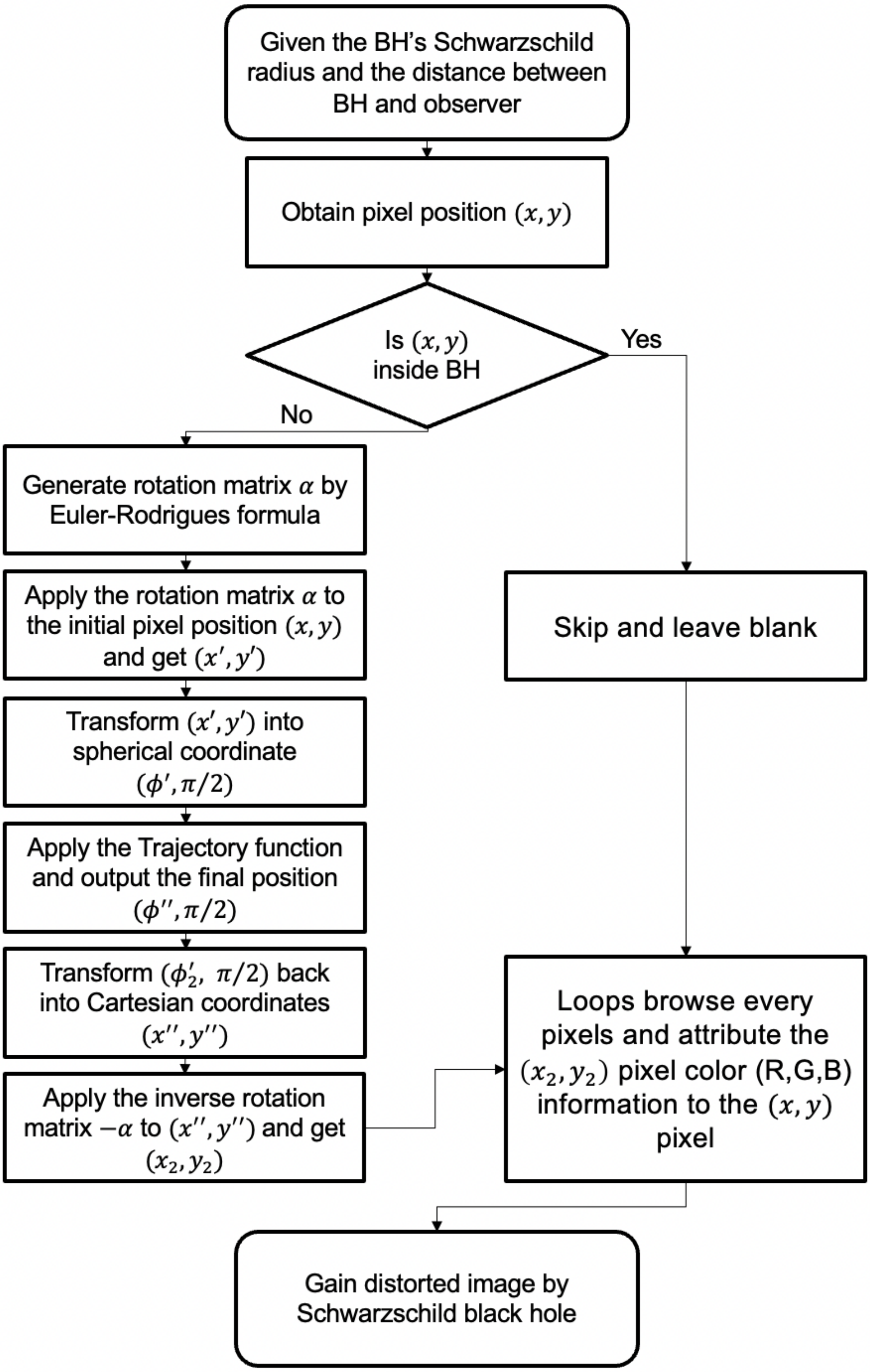


Figure 4.4 Coordinate Operations

4.2.2 Flowchart



4.2.3 Visualization Result

Write the code according to the flowchart above and use the default RGB image of the Python system as the input equirectangular image,

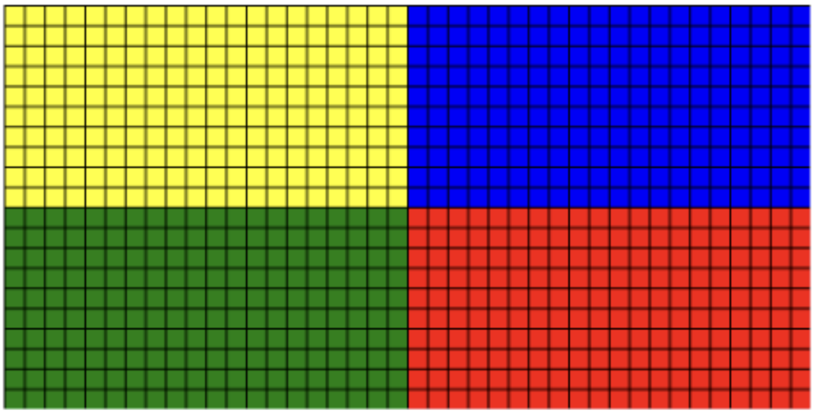


Figure 4.5 Default RGB Image

The equirectangular image, after being distorted by a Schwarzschild black hole with $Rs = 8$ and $D = 50$, is obtained as

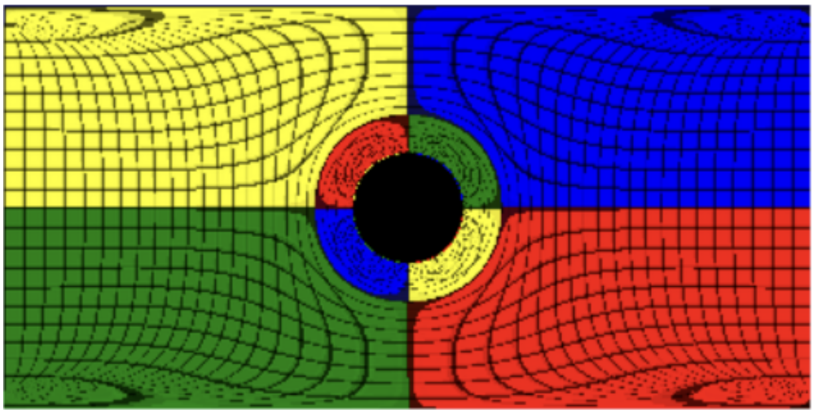


Figure 4.6 Image Distorted by a Schwarzschild Black Hole with $Rs = 8$, $D = 50$

Chapter 5

Conclusion

In this signature work, the basic concepts and definitions of differential geometry are briefly introduced, says metric, connection, Riemann curvature tensor and geodesic equation. Then, the components of Einstein's field equations are introduced according to these basic concepts and definitions.

Afterwards, the derivation of the Schwarzschild Metric is shown. Next, the concept of null geodesic is presented and the geodesics governed by the Schwarzschild Metric is derived using Euler-Lagrange equation. Later, event horizon and photon sphere are introduced as the properties derived from Schwarzschild Metric are shown.

As for creation parts, the trajectories of light rays near a Schwarzschild black hole are calculated by solving the geodesic equation in one variable numerically. Then the optical distortion by a Schwarzschild black hole of two-dimensional images can also be generated thanks to the circular symmetry. In order to give a better visualization of Schwarzschild black hole, a whole set of more complex code involving has been built.

This signature work could serve as starting point for modelling gravitational lensing of accretion disks around black holes.

Bibliography

[1] John Stewart, *Advanced General Relativity*, Cambridge University Press **4** (1991), 1-55.

[2] Kazunori Akiyama et al., *First M87 Event Horizon Telescope Results. I. The Shadow of the Supermassive Black Hole*, Astrophys (2019), J. 875(1), L1.

[3] ______, *First M87 Event Horizon Telescope Results. IV. Imaging the Central Supermassive Black Hole*, Astrophys (2019), J. 875(1), L4.

[4] ______, *First M87 Event Horizon Telescope Results. V. Physical Origin of the Asymmetric Ring*, Astrophys (2019), J. 875(1), L5.

[5] Subrahmanyan Chandrasekhar, *The Mathematical Theory of Black Holes*, Oxford University Press (1983), 1-55. ISBN 0-19-851291-0.